STEP Solutions
2011

Mathematics
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9470 STEP II 2011

Q1 **(i)** There are several routine features of a graph that one should look to consider on any curve-sketching question: key points, such as where the curve meets or cuts either of the coordinate axes, symmetries (and periodicities for trig. functions), asymptotes, and turning-points are the usual suspects. In this case, the given function involves square-roots as well, so the question of the domain of the function also comes into question. Considering all such things for $y = \sqrt{1-x} + \sqrt{3+x}$ should help you realise the following:

* the RHS is only defined for $-3 \le x \le 1$ (so the endpoints are at (–3, 2) and (1, 2));
* the graph is symmetric in the line $x = -1$, with its maximum at $\left(-1,\ 2\sqrt{2}\right)$**;** NB it must be a maximum since $2\sqrt{2} > 2$ so there is no need to resort to calculus to establish this;
* the curve is thus $\cap$ –shaped, and the gradient at the endpoints is infinite. This last point wasn't of great concern for the purposes of the question, so its mention was neither rewarded nor its lack penalised: however, this is easily determined by realising that each term in the RHS is of the form $X^{\frac{1}{2}}$, so their derivatives will be of the form $X^{-\frac{1}{2}}$ which, when evaluated at an endpoint will give one of them of the form $\frac{1}{0}$ symptomatic of an asymptote.

A quick sketch of $y = x + 1$ shows that there is only the one solution at $x = 1$.

(ii) Each side of this second equation represents an easily sketchable curve. Indeed, the RHS is essentially the same curve as in (i), but defined on the interval [–3, 3]. The LHS is merely a "horizontal" parabola, though only its upper half since the radix $\left(\sqrt{\ }\right)$ sign denotes the *positive* square-root. These curves again intersect only the once, when $x < 0$. Resorting to algebra … squaring, rearranging suitably and squaring again then yields a quadratic equation in x having one positive and one negative root.

Q2 The required list of perfect cubes is 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, though there were no marks for noting them.

(i) In this question, it is clearly important to be able to factorise the sum of two cubes. So, in this first instance $x + y = k$, $(x + y)(x^2 - xy + y^2) = kz^3 \Rightarrow \quad x^2 - (k-x)x + (k-x)^2 - z^3 = 0$, which gives the required result upon rearrangement. One could either treat this as a quadratic in x and deal with its discriminant or go ahead directly to show that $\frac{4z^3 - k^2}{3} = (y-x)^2 \ge 0$ which immediately gives that $\frac{4z^3 - k^2}{3}$ is a perfect square and also that $z^3 \ge \frac{1}{4}k^2$; and the other half of the required inequality comes either from $z^3 = k^2 - 3xy < k^2$ (since x, $y > 0$) or from noting that the smaller root of the quadratic in x is positive. Substituting $k = 20$ into the given inequality then yields $100 \le z^3 < 400 \Rightarrow z = 5, 6, 7$; and the only value of z in this list for which $\frac{4z^3 - k^2}{3}$ is a perfect square is $z = 7$, which then yields the solution $(x, y, z) = (1, 19, 7)$. Although not a part of the question, we can now express 20 as a sum of two rational cubes in the following way:

$$20 = \left(\frac{1}{7}\right)^3 + \left(\frac{19}{7}\right)^3 .$$

(ii) Although this second part of the question can be done in other ways, the intention is clearly that a similar methodology to (i)'s can be employed. Starting from

$$x + y = z^2,\ (x + y)(x^2 - xy + y^2) = kz.z^2 \Rightarrow x^2 - \left(z^2 - x\right)x + \left(z^2 - x\right)^2 - kz = 0$$

we find that $\frac{4kz - z^4}{3}$ is a perfect square, and also that $k < z^3 \le 4k$. With k = 19, $19 < z^3 \le 76$ $\Rightarrow$ z = 3 or 4. This time, each of these values of z gives $\frac{z(76 - z^3)}{3}$ a perfect square, yielding the two solutions (x, y, z) = (1, 8, 3) and (6, 10, 4). Thus we have two ways to represent 19 as a sum of two rational cubes: 19 = $\left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3$ and $\left(\frac{3}{2}\right)^3 + \left(\frac{5}{2}\right)^3$.

Purely as an aside, interested students may like to explore other possibilities for $x^3 + y^3 = kz^3$. One that never made it into the question was

$$x + y = kz,\ (x + y)(x^2 - xy + y^2) = kz.z^2 \ \Rightarrow\ x^2 - (kz - x)x + (kz - x)^2 - z^2 = 0$$

$\Rightarrow 3x^2 - 3kzx + z^2(k^2 - 1) = 0$. Then $x = \frac{3kz \pm \sqrt{9k^2z^2 - 12z^2(k^2 - 1)}}{6} = \frac{1}{6}z\left\{3k \pm \sqrt{12 - 3k^2}\right\}$, requiring $12 - 3k^2 \ge 0$ i.e. $k^2 \le 4 \Rightarrow k = 1$ or 2.

When k = 1: $x^3 + y^3 = z^3$ has NO solutions by *Fermat's Last Theorem*;

and when k = 2: $x^3 + y^3 = 2z^3$ has (trivially) infinitely many solutions $x = y = z$.

Q3 This question is all about increasing functions and what can be deduced from them. It involves inequalities, which are never popular creatures even amongst STEP candidates. Fortunately, you are led fairly gently by the hand into what to do, at least to begin with.

(i) (a) $f'(x) = \cos x - \{x.{-\sin x} + \cos x\} = x \sin x \ge 0$ for $x \in [0, \frac{1}{2}\pi]$, and since f(0) = 0 it follows that $f(x) = \sin x - x \cos x \ge 0$ for $0 \le x \le \frac{1}{2}\pi$.

(i) (b) A key observation here is that the "1" is simply a disguise for $\frac{d}{dx}(x)$, so you are actually being given that $\frac{d}{dx}(\arcsin x) \ge \frac{d}{dx}(x)$ in the given interval; in other words, that $f(x) = \arcsin x - x$ is an increasing function. Since f(0) = 0 and f increasing, $f(x) = \arcsin x - x \ge 0$ for $0 \le x < 1$, and the required result follows.

(i) (c) Writing $g(x) = \frac{x}{\sin x} \Rightarrow g'(x) = \frac{\sin x - x\cos x}{\sin^2 x} > 0$ for $0 < x < \frac{1}{2}\pi$ using (a)'s result. Now, it may help to write $u = \arcsin x$, just so that it looks simpler to deal with here. Then $u \ge x$ by (b)'s result $\Rightarrow g(u) \ge g(x)$ since $g'(x) \ge 0$ and the required result again follows.

(ii) There is a bit more work to be done here, but essentially the idea is the same as that in part (i), only the direction of the inequality seems to be reversed, so care must be taken. An added difficulty also arises in that we find that we must show that $f' \ge 0$ by showing that it is increasing from zero. So $g(x) = \frac{\tan x}{x}$, $g'(x) = \frac{x\sec^2 x - \tan x}{x^2} = \frac{2x - \sin 2x}{2x^2\cos^2 x}$. Examining $f(x) = 2x - \sin 2x$ (since the denominator is clearly positive in the required interval): f(0) = 0 and $f'(x) = 2 - 2\cos 2x \ge 0$ for $0 < x < \frac{1}{2}\pi \Rightarrow f \ge 0 \Rightarrow g'(x) \ge 0 \Rightarrow$ g increasing. Mimicking the conclusion of (i) (c), the reader should now be able to complete the solution.

Q4 **(i)** Using $\sin A = \cos(90^o - A)$ gives $\theta = 360n \pm (90^o - 4\theta)$ – Note that you certainly should be aware of the periodicities of the basic trig. functions $\Rightarrow 5\theta = 360n + 90^o$ or $3\theta = 360n + 90^o$. These give either $\theta = 72n + 18^o \Rightarrow \theta = 18^o, 90^o, 162^o$ or $\theta = 120n + 30^o \Rightarrow \theta = 30^o, 150^o$.

Now using the double-angle formulae for sine (twice) and cosine, we have $c = 2.2sc.(1 - 2s^2)$. We can discount $c = 0$ for $\theta = 18^o$, so that $1 = 4s(1 - 2s^2)$ which gives the cubic equation in $s = \sin\theta$, $8s^3 - 4s + 1 = 0 \Rightarrow (2s - 1)(4s^2 + 2s - 1) = 0$. Again, we can discount $c = \frac{1}{2}$ for $\theta = 18^o$) which leaves us with $\sin 18^o$ the positive root (as 18^o is acute) from the two possible solutions of this quadratic; namely, $\sin 18^o = \dfrac{\sqrt{5}-1}{4}$.

(ii) Using the double-angle formula for sine, we have $4s^2 + 1 = 16s^2(1 - s^2) \Rightarrow 0 = 16s^4 - 12s^2 + 1$ $\Rightarrow s^2 = \dfrac{12 \pm \sqrt{80}}{32} = \dfrac{3 \pm \sqrt{5}}{8}$. At first, this may look like a problem, but bear in mind that we want it to be a perfect square. Proceeding with this in mind, $s^2 = \dfrac{6 \pm 2\sqrt{5}}{16} = \left(\dfrac{\sqrt{5} \pm 1}{4}\right)^2$ so that we have the four answers, $\sin x = \pm\left(\dfrac{\sqrt{5} \pm 1}{4}\right)$.

(iii) To make the connection between this part and the previous one requires nothing more than division by 4 to get $\sin^2 x + \frac{1}{4} = \sin^2 2x$, and the solution $x = 3\alpha = 18^o$, $5\alpha = 30^o \Rightarrow \alpha = 6^o$ immediately presents itself from part (ii). However, in order to **deduce** a second solution (noting that $\alpha = 45^o$ is easily seen to satisfy the given equation), it is important to be prepared to be a bit flexible and use your imagination. The other possible angles that are "related" to 18^o and might satisfy (ii)'s equation, can be looked-for, provided that $\sin 5\alpha = \pm\frac{1}{2}$ (and there are many possibilities here also). A little searching and/or thought reveals

$\sin x = -\left(\dfrac{\sqrt{5}-1}{4}\right) \Rightarrow 3\alpha = 180^o + 18^o = 198^o$ also works, since $5\alpha = 330^o$ has $\sin 5\alpha = -\frac{1}{2}$,

and the second acute answer is $\alpha = 66^o$.

Q5 The simplest way to do this is to realise that OA is the bisector of $\angle BOC$, so that A is on the diagonal OA' of parallelogram $OBA'C$ (in fact, since $OB = OC$, it is a rhombus) $\Rightarrow$ b + c = λa for some λ (giving the first part of the result). Also, as BC is perpendicular to OA, $(\mathbf{b} - \mathbf{c}) \cdot \mathbf{a} = 0$ $\Rightarrow (2\mathbf{b} - \lambda\mathbf{a}) \cdot \mathbf{a} = 0 \Rightarrow \lambda = 2\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{a} \bullet \mathbf{a}}\right)$.

Similarly (replacing **a** by **b** and **b** by **c** in the above), we have $\mathbf{d} = k\mathbf{b} - \mathbf{c}$ where $k = 2\left(\dfrac{\mathbf{b} \bullet \mathbf{c}}{\mathbf{b} \bullet \mathbf{b}}\right)$

$= 2\left(\dfrac{\mathbf{b} \bullet \lambda\mathbf{a} - \mathbf{b} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) = 2\lambda\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) - 2 \Rightarrow \mathbf{d} = \left(2\lambda\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) - 2\right)\mathbf{b} - (\lambda\mathbf{a} - \mathbf{b}) = \mu\mathbf{b} - \lambda\mathbf{a}$ where

$\mu = 2\lambda\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\mathbf{b} \bullet \mathbf{b}}\right) - 1$ or $4\left(\dfrac{[\mathbf{a} \bullet \mathbf{b}]^2}{[\mathbf{a} \bullet \mathbf{a}][\mathbf{b} \bullet \mathbf{b}]}\right) - 1$.

Now A, B and D are collinear if and only if $\overrightarrow{AD} = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$ is a multiple of $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ $\Leftrightarrow t(\mathbf{b} - \mathbf{a}) = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$ for some t ($\neq 0$).

Comparing coefficients of **a** and **b** then gives (t =) $\mu = \lambda + 1$.

In the case when $\lambda = -\frac{1}{2}$, $\mu = \frac{1}{2}$ and D is the midpoint of AB.

Finally, $\mu = \frac{1}{2} \Rightarrow \frac{1}{2} = 4\left(\frac{[\mathbf{a} \bullet \mathbf{b}]^2}{[\mathbf{a} \bullet \mathbf{a}][\mathbf{b} \bullet \mathbf{b}]}\right) - 1 = 4\left(\frac{\mathbf{a} \bullet \mathbf{b}}{ab}\right)^2 - 1$, and using the scalar product formula $\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{ab}$ gives $\cos\theta = -\sqrt{\frac{3}{8}}$. [Note that **a . b** has the same sign as λ.]

Q6 To begin with, it is essential to realise that the integrand of $I = \int [f'(x)]^2 [f(x)]^n \, dx$ must have its two components split up suitably so that integration by parts can be employed. Thus

$$I = \int [f'(x)] \times \{[f'(x)][f(x)]^n\} dx = f'(x) \times \frac{1}{n+1}[f(x)]^{n+1} - \int \left([f''(x)] \times \frac{1}{n+1}[f(x)]^{n+1} \right) dx.$$

Now (and not earlier) is the opportune moment to use the given relationship $f''(x) = kf(x)f'(x)$, so that $I = f'(x) \times \frac{1}{n+1}[f(x)]^{n+1} - \int \left(kf'(x) \times \frac{1}{n+1}[f(x)]^{n+2} \right) dx$, which is now directly integrable as $f'(x) \times \frac{1}{n+1}[f(x)]^{n+1} - \frac{1}{(n+1)(n+3)} \times k[f(x)]^{n+3}$ (+ C).

(i) For $f(x) = \tan x$, $f'(x) = \sec^2 x$ and $f''(x) = 2\sec^2 x \tan x = kf(x)f'(x)$ with $k = 2$.

Also, differentiating $I = \frac{\sec^2 x \tan^{n+1} x}{n+1} - \frac{2\tan^{n+3} x}{(n+1)(n+3)}$ gives

$$\frac{dI}{dx} = \frac{1}{n+1}\left(\sec^2 x.(n+1)\tan^n x.\sec^2 x + 2\sec x.\sec x \tan x.\tan^{n+1} x\right)$$
$$- \frac{1}{(n+1)(n+3)}\left(2(n+3)\tan^{n+2} x.\sec^2 x\right) = \sec^4 x \tan^n x = \left(f'(x)\right)^2 \times \left(f(x)\right)^n \text{ as required,}$$

although this could be verified in reverse using integration. Using this result directly in the first given integral is now relatively straightforward:

$$\int \frac{\sin^4 x}{\cos^8 x} dx = \int \sec^4 x \tan^4 x \, dx = \frac{\sec^2 x \tan^5 x}{5} - \frac{2\tan^7 x}{35} + C.$$

(ii) Hopefully, all this differentiating of sec and tan functions may have helped you identify the right sort of area to be searching for ideas with the second of the given integrals.

If $f(x) = \sec x + \tan x$, $f'(x) = \sec x \tan x + \sec^2 x = \sec x(\sec x + \tan x)$

and $f''(x) = \sec^2 x(\sec x + \tan x) + \sec x \tan x(\sec x + \tan x)$

$= \sec x\,(\sec x + \tan x)^2 = kf(x)f'(x)$ with $k = 1$.

Then $\int \sec^2 x(\sec x + \tan x)^6 dx = \int \{\sec x(\sec x + \tan x)\}^2 \times (\sec x + \tan x)^4 dx$

$$= \frac{\sec x(\sec x + \tan x)^6}{5} - \frac{(\sec x + \tan x)^7}{35} + C$$

Q7 **(i)** Once you have split each series into sums of powers of λ and μ separately, it becomes clear that you are merely dealing with GPs. Thus $\sum_{r=0}^{n} b_r = \left(1+\lambda+\lambda^2+\ldots+\lambda^n\right)-\left(1+\mu+\mu^2+\ldots+\mu^n\right)$

$= \frac{\lambda^{n+1}-1}{\lambda-1}-\frac{\mu^{n+1}-1}{\mu-1} = \frac{1}{\sqrt{2}}\left(\lambda^{n+1}-1+\mu^{n+1}-1\right)$, since $\lambda-1=\sqrt{2}$ and $\mu-1=-\sqrt{2}$

$= \frac{1}{\sqrt{2}}a_{n+1}-\sqrt{2}$ and, similarly, $\sum_{r=0}^{n} a_r = \frac{\lambda^{n+1}-1}{\sqrt{2}}-\frac{\mu^{n+1}-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}b_{n+1}$.

(ii) There is no need to be frightened by the appearance of the nested sums here as the 'inner sum' has already been computed: all that is left is to work with the remaining 'outer sum' and deal carefully with the limits: $\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \sum_{m=0}^{2n}\left(\frac{1}{\sqrt{2}}b_{m+1}\right) = \frac{1}{\sqrt{2}}\sum_{m=0}^{2n+1} b_m$ (since $b_0 = 0$)

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}a_{2n+2}-\sqrt{2}\right) = \frac{1}{2}\left(\lambda^{2n+2}+\mu^{2n+2}-2\right) = \frac{1}{2}\left(\left[\lambda^{n+1}\right]^2-2\left[\lambda\mu\right]^{n+1}+\left[\mu^{n+1}\right]^2\right)$ since $\lambda\mu = -1$

and n + 1 is even when n is odd $= \frac{1}{2}\left(b_{n+1}\right)^2$ when n is odd. However, when n is even, n + 1 is odd

and $\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \frac{1}{2}\left(b_{n+1}\right)^2-2$ or $\frac{1}{2}\left(a_{n+1}\right)^2$.

(iii) We already have the result $\left(\sum_{r=0}^{n} a_r\right)^2 = \frac{1}{2}\left(b_{n+1}\right)^2$, so the only new thing is

$\sum_{r=0}^{n} a_{2r+1} = \left(\lambda+\lambda^3+\lambda^5+\ldots+\lambda^{2n+1}\right)+\left(\mu+\mu^3+\mu^5+\ldots+\mu^{2n+1}\right)$, which is still the sum of two GPs, merely with different common ratios, having sum $\frac{\lambda\left(\lambda^{2n+2}-1\right)}{\lambda^2-1}+\frac{\mu\left(\mu^{2n+2}-1\right)}{\mu^2-1}$.

Now $\lambda^2-1 = 3+2\sqrt{2}-1 = 2\left(1+\sqrt{2}\right) = 2\lambda$ and $\mu^2-1 = 3-2\sqrt{2}-1 = 2\left(1-\sqrt{2}\right) = 2\mu$,

so $\sum_{r=0}^{n} a_{2r+1} = \frac{1}{2}\left(\lambda^{2n+2}+\mu^{2n+2}-2\right) = \frac{1}{2}\left(b_{n+1}\right)^2$ when n is odd, and $\frac{1}{2}\left(b_{n+1}\right)^2-2$ when n is even.

Thus $\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 0$ when n is odd / = 2 when n is even.

Q8 The string leaves the circle at $C(-\cos\theta, \sin\theta)$.
Since the radius of the circle is 1, Arc $AC = \pi - t = \theta$ (so $\cos\theta = -\cos t$ and $\sin\theta = \sin t$).
Then $B = (-\cos\theta + t\sin\theta, \sin\theta + t\cos\theta) = (\cos t + t\sin t, \sin t - t\cos t)$.

$\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t$ by the *Product Rule*; = 0 when $t = 0$, $(x, y) = (1, 0)$ or $t = \frac{1}{2}\pi$, $(x, y) = \left(\frac{1}{2}\pi, 1\right)$. This is x_{max} so $t_0 = \frac{1}{2}\pi$.

The required area under the curve and above the x-axis is

$$A = \int_{\pi}^{\frac{1}{2}\pi} y\frac{dx}{dt}\,dt = \int_{\pi}^{\frac{1}{2}\pi}\left(\sin t - t\cos t\right)t\cos t\,dt = \int_{\frac{1}{2}\pi}^{\pi} -\tfrac{1}{2}t\sin 2t\,dt + \int_{\frac{1}{2}\pi}^{\pi}\tfrac{1}{2}t^2\left(1+\cos 2t\right)\,dt$$

using the double-angle formulae for sine and cosine. As the integration here may get very messy, it is almost certainly best to evaluate this area as the sum of three separate integrals:

$$\int_{\frac{1}{2}\pi}^{\pi} -\frac{1}{2}t\sin 2t\,\mathrm{d}t = \left[\frac{1}{4}t\cos 2t\right]_{\frac{1}{2}\pi}^{\pi} - \int_{\frac{1}{2}\pi}^{\pi}\frac{1}{4}\cos 2t\,\mathrm{d}t = \left[\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t\right]_{\frac{1}{2}\pi}^{\pi} = \frac{3\pi}{8};$$

$$\int_{\frac{1}{2}\pi}^{\pi}\frac{1}{2}t^2\ \mathrm{d}t = \left[\frac{1}{6}t^3\right]_{\frac{1}{2}\pi}^{\pi} = \frac{7\pi^3}{48}$$

and $\int_{\frac{1}{2}\pi}^{\pi}\tfrac{1}{2}t^2\cos 2t\ \mathrm{d}t = \left[\frac{1}{4}t^2\sin 2t\right]_{\frac{1}{2}\pi}^{\pi} - \int_{\frac{1}{2}\pi}^{\pi}\frac{1}{2}t\sin 2t\,\mathrm{d}t = 0 - -\frac{3\pi}{8} = \frac{3\pi}{8}$ using a previous answer.

Thus $A = \dfrac{7\pi^3}{48} + \dfrac{3\pi}{4}$.

For the total area swept out by the string during this process (called *Involution*), we still need to add in the area swept out between $t = 0$ and $t = \tfrac{1}{2}\pi$, which is $-\dfrac{\pi^3}{48} + \dfrac{\pi}{4}$ (there is, of course, no need to repeat the integration process), and then subtract the area inside the semi-circle. Thus the total area swept out by the string is $\dfrac{7\pi^3}{48} + \dfrac{3\pi}{4} + \left(-\dfrac{\pi^3}{48} + \dfrac{\pi}{4}\right) - \dfrac{\pi}{2}$ (area inside semi-circle) = $\dfrac{\pi^3}{6}$.

Q9 Collisions questions are always popular, as there are only two or three principles which are to be applied. It is, nonetheless, good practice to say what you are attempting to do. Also, a diagram, though not an essential requirement, is almost always a good idea, if only since it enables you to specify a direction which you are going to take to be the positive one, especially since velocity and momentum are vector quantities. Once these preliminaries have been set up, the rest is fairly easy. By *CLM*, $3mu = 2mV_A + mV_B$ and *NEL/NLR* gives $e.3u = V_B - V_A$. Solving these simultaneously for V_A and V_B yields $V_A = u(1 - e)$ and $V_B = u(1 + 2e)$.

Next, after its collision with the wall, B has speed $|\,f\,V_B\,|$ away from the wall.

For the second collision of A and B, by *CLM* (away from wall), $fmV_B - 2mV_A = 2mW_A - mW_B$, and *NEL/NLR* gives $W_A + W_B = e(V_A + f\,V_B)$. Subst[g]. for V_A & V_B from before in *both* of these equations $\Rightarrow$ $2W_A - W_B = u\{f(1+2e) - 2(1-e)\}$ and $W_A + W_B = eu\{(1-e) + f(1+2e)\}$. Solving these simultaneously for W_A (not wanted) and W_B then gives $W_B = \tfrac{1}{3}u\{2(1-e^2) - f(1-4e^2)\}$, as required.

Noting that $1 - 4e^2$ can be negative, zero, or positive, it may be best (though not essential) to consider the possible cases separately:

if $e = \tfrac{1}{2}$, $W_B = \tfrac{1}{3}u\{2(\tfrac{3}{4}) - f(0)\} = \tfrac{1}{2}u > 0$;

if $\tfrac{1}{2} < e < 1$, $W_B = \tfrac{1}{3}u\{2(1-e^2) + f(4e^2 - 1)\} > 0$ for all e, f since each term in the bracket is > 0;

if $0 < e < \tfrac{1}{2}$, $1 - e^2 > \tfrac{3}{4}$ and $W_B > \tfrac{1}{3}u\{\tfrac{3}{2} - f(1 - 4e^2)\} > \tfrac{1}{3}u\{\tfrac{3}{2} - 1\times 1\} > 0$.

Q10 The maximum height of a projectile is when $\dot{y} = u\sin\theta - gt = 0 \Rightarrow t = \dfrac{u\sin\theta}{g}$. Substituting this into $y = ut\sin\theta - \tfrac{1}{2}gt^2 \Rightarrow H = \dfrac{u^2\sin^2\theta}{2g}$ (although some people learn it to quote).

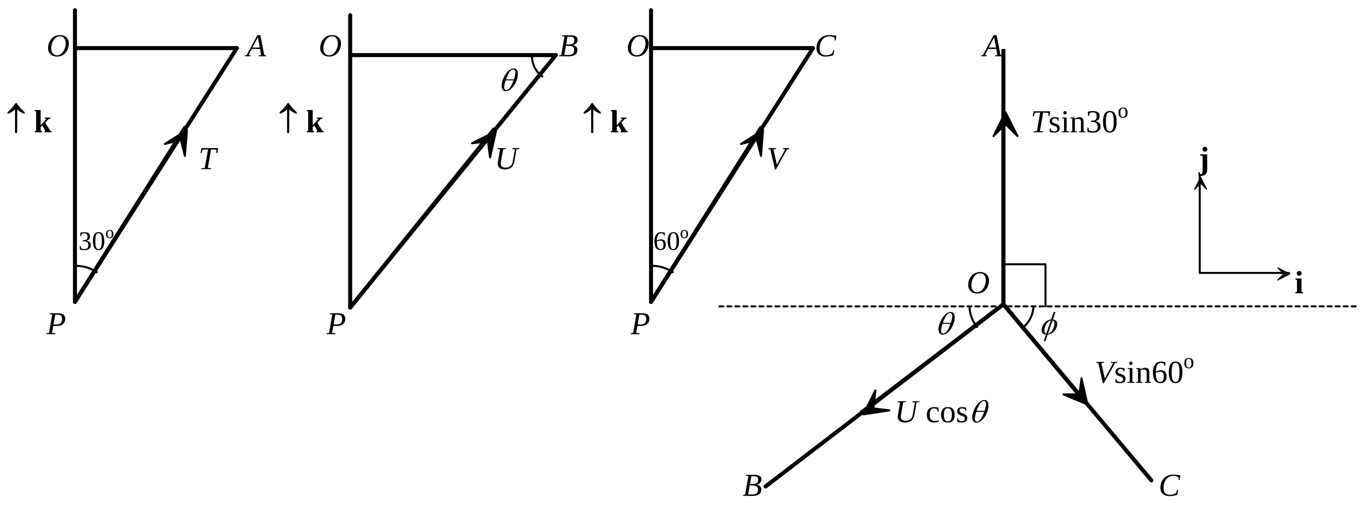
When the string goes taut, its length l is given by $l = \tfrac{1}{2}H = \dfrac{u^2\sin^2\theta}{4g}$. But l is also given from the y-component of P's displacement as $l = ut\sin\theta - \tfrac{1}{2}gt^2$, which gives the quadratic equation $gt^2 - (2u\sin\theta)t + H = 0$ in t. Solving by the quadratic formula, $t = \dfrac{2u\sin\theta \pm \sqrt{4u^2\sin^2\theta - 4gH}}{2g}$ $= \dfrac{2\sqrt{2gH} \pm \sqrt{8gH - 4gH}}{2g} = \dfrac{1}{g}\left(\sqrt{2gH} \pm \sqrt{gH}\right) = \sqrt{\dfrac{H}{g}}\left(\sqrt{2} - 1\right)$, where we take the smaller of the two roots since we want the first time when an unimpeded P is at this height.

At this time, P's vertical velocity is $v = \dot{y} = u\sin\theta - g\sqrt{\dfrac{H}{g}}\left(\sqrt{2}-1\right) = \sqrt{2gH} - \sqrt{gH}\left(\sqrt{2}-1\right)$ $= \sqrt{gH}$ or $\dfrac{u\sin\theta}{\sqrt{2}}$. Thus, the common speed of P/R after the string goes taut, by *CLM*, is $\tfrac{1}{2}\sqrt{gH}$ or $\dfrac{u\sin\theta}{2\sqrt{2}}$.

When the string goes slack, we must consider the projectile motion of R, which has initial velocity components $u\cos\theta \rightarrow$ and $\dfrac{u\sin\theta}{2\sqrt{2}}\uparrow$. [Note that both P and R move in this way, so P no longer interferes with R's motion.] R's vertical displacement is zero when $y_R = \dfrac{u\sin\theta}{2\sqrt{2}}t - \dfrac{g}{2}t^2 = 0$ $(t \neq 0)$ $\Rightarrow t = \dfrac{u\sin\theta}{g\sqrt{2}}$ (and this is the extra time after the string has gone slack). The total distance travelled by R is thus $D = x_1 + x_2$, where $x_1 = u\cos\theta\dfrac{u\sin\theta}{g\sqrt{2}}\left(\sqrt{2}-1\right)$ and $x_2 = u\cos\theta\dfrac{u\sin\theta}{g\sqrt{2}}$ $= \dfrac{u^2\sin\theta\cos\theta}{g}$.

Finally, setting $D = H \Rightarrow \tan\theta = 2$.

Q11 **(i)** The saying goes that "a picture paints a thousand words" and tis is especially true in mechanics questions, if for no better reason than it gives the solver a clear indication of angles/directions for – in this case – the forces involved. The relevant diagrams are as follows:



It might also be wise to note the sines and cosines of the given angles: $\tan\theta = \sqrt{2} \Rightarrow \sin\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \dfrac{1}{\sqrt{3}}$, and $\tan\phi = \dfrac{\sqrt{2}}{4} \Rightarrow \sin\phi = \dfrac{1}{3}$ and $\cos\phi = \dfrac{2\sqrt{2}}{3}$. Having noted these carefully, it is now reasonably straightforward to state that the vector in the direction of PB is

$$-(U\cos\theta).\cos\theta\,\mathbf{i} - (U\cos\theta).\sin\theta\,\mathbf{j} + U\sin\theta\,\mathbf{k} \;=\; -\frac{1}{3}\mathbf{i} - \frac{\sqrt{2}}{3}\mathbf{j} + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{k}\,.$$

Note that the question requires you to verify that this vector has magnitude 1.

(ii) The forces involved are now readily written down …

$\underline{\boldsymbol{T}}_B \;=\; \left(-\dfrac{1}{3}\mathbf{i} - \dfrac{\sqrt{2}}{3}\mathbf{j} + \dfrac{\sqrt{2}}{\sqrt{3}}\mathbf{k}\right)U$ follows from (i)'s answer. Also,

$$\underline{\boldsymbol{T}}_A \;=\; T\sin 30°\,\mathbf{j} + T\cos 30°\,\mathbf{k} \;=\; \frac{1}{2}T\left(\mathbf{j} + \sqrt{3}\,\mathbf{k}\right),$$

$$\underline{\boldsymbol{T}}_C \;=\; V\sin 60°\cos\phi\,\mathbf{i} - V\sin 60°\sin\phi\,\mathbf{j} + V\cos 60°\,\mathbf{k} \;=\; \frac{1}{2}V\left(\frac{2\sqrt{2}}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \mathbf{k}\right)$$

and $\underline{\boldsymbol{W}} = -\,W\,\mathbf{k}$.

(iii) Having set the system up in vector form, the fundamental Statics principle involved is that

$$\underline{\boldsymbol{T}}_A + \underline{\boldsymbol{T}}_B + \underline{\boldsymbol{T}}_C + \underline{\boldsymbol{W}} = \underline{\mathbf{0}}\,.$$

Comparing components in this vector equation gives

(**i**) $0 - \dfrac{1}{3}U + \dfrac{\sqrt{6}}{3}V = 0 \Rightarrow U = V\sqrt{6}$

(**j**) $\dfrac{1}{2}T - \dfrac{\sqrt{2}}{3}U - \dfrac{\sqrt{3}}{6}V = 0 \Rightarrow$ (using $U = V\sqrt{6}$) $T = \dfrac{5\sqrt{3}}{3}V$

(**k**) $\dfrac{\sqrt{3}}{2}T + \dfrac{\sqrt{6}}{3}U + \dfrac{1}{2}V = W \Rightarrow$ (using $U = V\sqrt{6}$ and $T = \dfrac{5\sqrt{3}}{3}V$)

$$T = \frac{W\sqrt{3}}{3},\;\; U = \frac{W\sqrt{6}}{5},\;\; V = \frac{W}{5}.$$

Q12 It is important in these sorts of contrived games to read the rules properly: in this case, you must ensure that you are clear what is meant by 'match', 'game' and 'point'. Then, a careful listing of cases is all that is required.

(i) P(re-match) = P(XYX) + P(YXY) = $p(1-p)^2 + (1-p)^3 = (1-p)^2$.

P(Y wins directly) = P(YY) + P(XYY) = $(1-p)p + p(1-p)p = p(1-p)(1+p)$ or $p\left(1-p^2\right)$.

Thus, P(Y wins) = w = $p\left(1-p^2\right) + w(1-p)^2$, and you will note the way starting the match again leads to a recurrent way of describing Y's chances of winning. Re-arranging this then gives

$$w = \frac{p\left(1-p^2\right)}{1-(1-p)^2} = \frac{p\left(1-p^2\right)}{\left(1-(1-p)\right)\left(1+(1-p)\right)} = \frac{p\left(1-p^2\right)}{p(2-p)} = \frac{1-p^2}{2-p} \quad \text{for } p \neq 0.$$

Next, $w - \frac{1}{2} = \frac{2\left(1-p^2\right)-(2-p)}{2(2-p)} = \frac{p(1-2p)}{2(2-p)}$, and since $2-p > 0$, $w - \frac{1}{2}$ has the same sign as $1-2p$ and hence as $\frac{1}{2} - p$. Hence, $w > \frac{1}{2}$ if $p < \frac{1}{2}$ and $w < \frac{1}{2}$ if $p > \frac{1}{2}$.

To be fair at this point, the final demand of part (i) ended up being rather less demanding than was originally intended, as the answer is either "Yes" or "No" … though you would of course, be expected to support your decision; no marks are given for being a lucky guesser! The following calculus approach is thus slightly unnecessary, as one can simply provide an example to show that w can decrease with p. The following, more detailed analysis had been intended.

$$\frac{dw}{dp} = \frac{(2-p)(-2p)-\left(1-p^2\right)(-1)}{(2-p)^2} = \frac{1}{(2-p)^2}\left(p^2-4p+1\right) = \frac{1}{(2-p)^2}\left([2-p]^2-3\right).$$

Then $\frac{dw}{dp} > 0$ for $0 < p < 2-\sqrt{3}$ and $\frac{dw}{dp} < 0$ for $2-\sqrt{3} < p \leq 1$.

For a fair game, Y's expectation should be 0. Thus, using E(gain) = $\sum g_i \times \mathrm{P}(g_i)$, where g_i is the "gain function" for Y, with $w = \frac{5}{12}$ when $p = \frac{2}{3}$, we have $0 = (k)\times\frac{5}{12} + (-1)\times\frac{7}{12} \Rightarrow k = 1.4$.

When $p = 0$, the results run YXY ... re-match ... YXY ... re-match ... and the match never ends.

Q13 Firstly, *skewness* is a measure of a distribution's **lack** of symmetry.

(i) For the next part, you should understand how the "expectation" function behaves.

$$E\left[(X-\mu)^3\right] = E\left[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right] = E\left[X^3\right] - 3\mu E\left[X^2\right] + 3\mu^2 E[X] - \mu^3$$

$$= E\left[X^3\right] - 3\mu\left(\sigma^2+\mu^2\right) + 3\mu^2.\mu - \mu^3 \quad \text{using } \mathrm{E}[X] = \mu \text{ and } \mathrm{E}[X^2] = \sigma^2 + \mu^2$$

$$= E\left[X^3\right] - 3\mu\sigma^2 - \mu^3, \text{ as required.}$$

For a given distribution, this next bit of work is very routine indeed.

$E[X] = \int_0^1 2x^2\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3} = \mu$; $E\left[X^2\right] = \int_0^1 2x^3\,dx = \left[\frac{1}{2}x^4\right]_0^1 = \frac{1}{2} \Rightarrow \sigma^2 = \frac{1}{18}$; and

$E\left[X^3\right] = \int_0^1 2x^4\,dx = \left[\frac{2}{5}x^5\right]_0^1 = \frac{2}{5}$; all of which then lead to $\gamma = \dfrac{\frac{2}{5} - 3.\frac{2}{3}.\frac{1}{18} - \frac{8}{27}}{\frac{1}{18\sqrt{18}}} = -\dfrac{2\sqrt{2}}{5}$ when substituted into the given (previously deduced) formula.

(ii) Here, $F(x) = \int_0^x 2x\,dx = x^2$ $(0 \le x \le 1) \Rightarrow F^{-1}(x) = \sqrt{x}$ $(0 \le x \le 1)$

$$\Rightarrow D = \frac{F^{-1}\left(\frac{9}{10}\right) - 2F^{-1}\left(\frac{1}{2}\right) + F^{-1}\left(\frac{1}{10}\right)}{F^{-1}\left(\frac{9}{10}\right) - F^{-1}\left(\frac{1}{10}\right)} = \frac{\frac{3}{\sqrt{10}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}}} = \frac{3 - 2\sqrt{5} + 1}{3 - 1} = \frac{4 - 2\sqrt{5}}{2} = 2 - \sqrt{5}.$$

M is given by $\int_0^M 2x\,dx = \frac{1}{2} \Rightarrow M^2 = \frac{1}{2} \Rightarrow M = \frac{1}{\sqrt{2}}$ (OR by $M = F^{-1}(\frac{1}{2}) = \frac{1}{\sqrt{2}}$).

And $P = \dfrac{3\left(\frac{2}{3} - \frac{1}{\sqrt{2}}\right)}{\frac{1}{3\sqrt{2}}} = 6\sqrt{2} - 9$.

In order to establish the given inequality "chain", we must show that $D > P$ and $P > \gamma$ (there is no point in proving that $D > \gamma$). One could reason this through by considering approximants to $\sqrt{2}$ and $\sqrt{5}$, but care must be taken not to introduce fallacious "roundings" which don't support the direction of the inequality under consideration. The alternative is to establish a set of equivalent numerical statements; for example, to show that $D > P$ …

$2 - \sqrt{5} > 6\sqrt{2} - 9 \Leftrightarrow 11 - \sqrt{5} > 6\sqrt{2}$

$\Leftrightarrow 121 + 5 - 22\sqrt{5} > 72$ (after squaring, since both sides are positive)

$\Leftrightarrow 54 > 22\sqrt{5}$ or $27 > 11\sqrt{5} \Leftrightarrow 729 > 605$ (again, squaring positive terms)

and this final result clearly *IS* true, so the desired inequality is established.